Asymptotic expansion for renewal functions, with application.

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Framework

$(X_k)_{k\in\mathbb{N}}$ i.i.d. ≥ 0 with c.d.f. $F(.)$.

Renewal process N defined by

$$N(x) := \sup\left\{ n \geq 0 \,\middle|\, S_n := \sum_{j=1}^{n} X_j \leq x \right\}, \quad x \geq 0,$$

and associated renewal function

$$\boxed{U(x) := \mathbb{E}[N(x)] = \sum_{n=1}^{\infty} \mathbb{P}[S_n \leq x] = \sum_{n=1}^{\infty} F^{*(n)}(x), \quad x \geq 0.}$$

Framework

$U(x)$ = mean number of occurances of a certain recurrent event before time x.

$\longrightarrow$ behaviour as $x \to \infty$?

In what follows, two cases :

- X_k's lattice ($\iff$ X_k with values in $d\mathbb{N}$ for some d, $d = 1$ onward),
- X_k's non lattice.

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Expansion of order 1

If X_1 admits a first moment μ then Blackwell's "elementary" renewal theorem $\Longrightarrow$

$$U(x+h) - U(x) \longrightarrow \frac{h}{\mu}, \quad x \to +\infty,\ x \in \mathbb{R}_+,\ h > 0, \quad \text{(non lattice)},$$

$$U(k+1) - U(k) \longrightarrow \frac{1}{\mu}, \quad k \to +\infty,\ k \in \mathbb{N}, \quad \text{(lattice)}.$$

Hence first order expansion :

$$\boxed{U(x) \sim \frac{x}{\mu}, \quad x \to \infty}$$

Expansion of order 2

If X_1 admits a second moment $\mu_2 = \mathbb{E}[X_1^2]$ then second order expansion (e.g. Feller (1965))

$$\boxed{U(x) = \begin{cases} \dfrac{x}{\mu} + \dfrac{\mu_2}{2\mu^2} + o(1), & \text{non lattice} \\ \dfrac{x}{\mu} + \dfrac{\mu_2 + \mu}{2\mu^2} + o(1), & \text{lattice.} \end{cases} \quad \text{as } x \to \infty,}$$

One even has $U(x) - \frac{x}{\mu} \geq 0$, $\forall x \geq 0$.

The $o(1)$ term

We set $\boxed{v(x) := U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2}.}$ (non lattice)

$\longrightarrow$ Behavior of $v(x)$ as $x \to \infty$?

- Stone (1965) : in the case where X_1 is light tailed ($\iff \mathbb{E}[e^{R_0 X_1}] < +\infty$ for some $R_0 \in (0, +\infty]$) then

$$v(x) = O(e^{-rx}), \quad x \to +\infty$$

  for some $r > 0$.

- Asmussen (1995) : in the case where X_1 has rational Laplace Transform then Explicit expression of $v(x)$ (i.e. of $U(x)$).
- Mitov and Omey (2014) provide intuitive approximations of $U(x)$, and in particular of the $v(x)$ term, for a large class of X_1.

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Introduction and Notation

Expansion of order 1 and 2

Expansion of order N, light tailed case

Application to a replacement model

The result, non lattice case

Theorem (Dombry, R. (2014))

Let us suppose that X_1 is non lattice, light tailed with $\mathbb{E}[e^{R_0 X_1}] < +\infty$, and satisfies the following assumption :

(A) *the equation $g(z) := \mathbb{E}[e^{zX_1}] = 1$ has a finite number of solutions in $S_{R_0} = \{z \in \mathbb{C}, 0 < \Re(z) < R_0\}$.*

Let $z_0 = 0, z_1, \ldots, z_N$ be these solutions. Then, for all $r < R_0$,

$$v(x) = \sum_{j=1}^{N} \rho_j e^{-x\Re(z_j)} \cos(x\Im(z_j) + \varphi_j) + o(e^{-rx}), \quad \text{as } x \to +\infty,$$

In the case $g'(z_j) \neq 0$, ρ_j and $\varphi_j \in (-\pi, \pi]$ are such that $\rho_j e^{i\varphi_j} = \frac{1}{z_j g'(z_j)}$.

The result, lattice case

Theorem (Dombry, R. (2014), Ct'd)

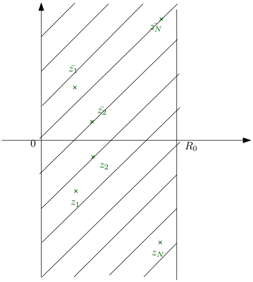
Let us suppose that X_1 is lattice, light tailed $R_0 \in (0, +\infty]$. Let $z_0 = 0, z_1, \ldots, z_N$ the solutions of the equation $g(z) := \mathbb{E}[e^{zX_1}] = 1$ in the domain $S_{R_0} = \{z \in \mathbb{C}; 0 < \Re(z) < R_0, -\pi \leq \Im(z) \leq \pi\}$. Then, for all $r < R_0$, $v(k)$ has the asymptotic expansion

$$v(k) = \sum_{j=1}^{N} \rho_j e^{-k\Re(z_j)} \cos(k\Im(z_j) + \varphi_j) + o(e^{-rk}), \quad k \to +\infty, k \in \mathbb{N},$$

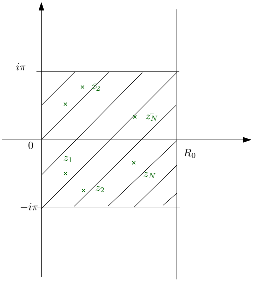
In the case $g'(z_j) \neq 0$, ρ_j and $\varphi_j \in (-\pi, \pi]$ are such that $\rho_j e^{i\varphi_j} = \frac{1}{(e^{z_j}-1)g'(z_j)}$.

Prior comments

Main practical issue is solve $g(z) = 1$ with $g(z) := \mathbb{E}[e^{zX_1}]$



S_{R_0} domain, non lattice



S_{R_0} domain, lattice

No trivial solution in $\mathbb{C}$ (except $z = 0$).
E.g. $X_1 \sim \mathcal{U}([0,1])$ we get to solve

$$e^z = z + 1, \quad z \in \mathbb{C}.$$

Elements of Proof (lattice case), Stone (1965) revisited

Recall that $v(k) := U(k) - \frac{k}{\mu} - \frac{\mu_2+\mu}{2\mu^2}$ and that X_1 concentrated on $\mathbb{N}$.

Set $S_n = \sum_{j=1}^{n} X_j$, $S_0 = 0$, and

$$\boxed{u_k := \sum_{n=0}^{\infty} \mathbb{P}[S_n = k] = U(k) - U(k-1), \quad k \in \mathbb{N}.}$$

Step 1 : one proves that

$$u_k - \frac{1}{\mu} = \frac{1}{2\mu} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re\left(e^{-ik\theta} \left[\frac{1}{1-g(i\theta)} - \frac{1}{\mu} \frac{1}{1-e^{i\theta}} \right] \right) d\theta$$

(recall that $g(i\theta) = \mathbb{E}[e^{i\theta X_1}]$)

Elements of Proof (lattice case), Stone (1965) revisited

Step 2 : Integrate $z \mapsto \frac{1}{1-\mathbb{E}[e^{i\theta X_1}]} - \frac{1}{\mu}\frac{1}{1-e^{i\theta}}$ on contour ∂S_r for $r < R_0$ and use Theorem of Residue in order to get

$$u_k - \frac{1}{\mu} = -\sum_{j=1}^{N} \Re\left[\frac{e^{-kz_j}}{g'(z_j)}\right] + o(e^{-rk})$$

(in the case $g'(z_j) \neq 0$, for ease of presentation...).

Step 3 : use the fact that

$$v(k) = \sum_{m=0}^{\infty}[-v(k+m+1) + v(k+m)] = \sum_{m=0}^{\infty}[-u_{k+m+1} + 1/\mu]$$

then conclude.

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The (simple) model

- Component with generic lifetime distribution L
- Replaced at each failure time with new component with probability $p \in (0, 1)$.

Total Lifetime : $\boxed{T = \sum_{k=1}^{\nu} L_k}$, where $L_1, L_2, \ldots$ i.i.d. and $\nu \sim \mathcal{G}(1 - p)$.

Laplace Transform of T, $\mathbb{E}[T]$, $\mathrm{Var}(T)$ computable, what about survival function ?

Estimate for lifetime survival function

Set

$$\boxed{\bar{H}(x) := \mathbb{P}[T > x] = \mathbb{P}\left[\sum_{k=1}^{\nu} L_k > x\right]}$$

$\longrightarrow$ Expansion of $\bar{H}(x)$ as $x \to \infty$?

Main Assumption :

- L bounded by some $M > 0$,
- density $f(x)$ of L is decreasing (e.g. holds if DFR).

Estimate for lifetime survival function

In that case we have the expansion for some R large enough

$$\boxed{\bar{H}(x) = \sum_{j=1}^{N} \Re\left[\frac{1-1/p}{1/p - f(0+)\mathbb{E}[Ze^{z_j Z}]} e^{-xz_j}\right] + o(e^{-rx}), \quad \forall r > R,}$$

where Z is a r.v. with cdf $\mathbb{P}[Z \leq x] = 1 - \dfrac{f(x)}{f(0+)}$ and $z_1, \ldots, z_N$ roots of Equation

$$1 + \frac{z}{f(0+)p} = \mathbb{E}[e^{zZ}], \quad z \in \mathbb{C},$$

with positive real part.

Thank you !